# Quantum cloning without signaling

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**Abstract**

Perfect Quantum Cloning Machines (QCM) would allow to use quantum nonlocality for arbitrary fast signaling. However perfect QCM cannot exist. We derive a bound on the fidelity of QCM compatible with the no-signaling constraint. This bound equals the fidelity of the Bužek-Hillery QCM.

Quantum mechanics is non-local, but this cannot be used to signal. Indeed, the different mixtures of pure states that can be prepared at a distance cannot be distinguished. For example, if two distant spin $\frac{1}{2}$ particle are in the singlet state, then a measurement of $\vec{m}\vec{\sigma}$ on one of them results in a mixture for the second particle spin state: $\pm\vec{m}$ on the Bloch sphere, with the same probability $\frac{1}{2}$ for both signs. This mixture corresponds to the density matrix $\frac{1}{2}\mathbb{1}$, independently of the measurement direction $\vec{m}$. If one could, in some way or another, distinguish between different mixtures that can be prepared at a distance, then quantum non locality could be used for signaling and the peaceful coexistence between quantum mechanics and relativity[1] would be broken. If, for example, perfect quantum cloning machines (QCM) would exists, then, by cloning the second particle, the mixtures corresponding to different directions $\vec{m}$ could be distinguished. But perfect QCM do not exist[2]. However, imperfect QCM exist[3, 4, 5]. In this letter we derive the upper bound of the quality (fidelity) of QCM of qubits (spin $\frac{1}{2}$) compatible with no signaling: any hypothetical QCM with higher fidelity would allows signaling.

Let $\rho_{in}(\vec{m}) = \frac{1+\vec{m}\vec{\sigma}}{2}$ and $\rho_{out}(\vec{m})$ denote the input and the corresponding output state of our hypothetical QCM, with $\rho_{out}(\vec{m})$ a 2-qubit state. We assume that the 2 output qubits are in the same state: $Tr_1(\rho_{out}(\vec{m})) = Tr_2(\rho_{out}(\vec{m})) = \frac{1+\eta\vec{m}\vec{\sigma}}{2}$, where $\eta$ is the shrinking factor of the Bloch vector $\vec{m}$. The cloning fidelity is given by: $\mathcal{F} \equiv Tr(\rho_{in}(\vec{m})\frac{1+\eta\vec{m}\vec{\sigma}}{2}) = \frac{1+\eta}{2}$. Let us emphasize that we do not assume any further particular relation between $\rho_{in}(\vec{m})$ and $\rho_{out}(\vec{m})$, in particular we do not assume any linear relation. In full generality, the output state can be written as:

$$\rho_{out}(\vec{m}) = \frac{1}{4}(\mathbb{1} + \eta(\vec{m}\vec{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\vec{m}\vec{\sigma}) + \sum_{j,k=x,y,z} t_{jk}\sigma_j\otimes\sigma_k) \tag{1}$$

By definition, universal QCM act similarly on all input states:

$$\rho_{out}(U\vec{m}) = U\otimes U\rho_{out}(\vec{m})U^\dagger\otimes U^\dagger \tag{2}$$

for all $U$, where the same notation $U$ is used for unitary operators acting on the 2-dimensional spin $\frac{1}{2}$ Hilbert space and for the corresponding rotation operator acting on the Bloch vectors $\vec{m}$. A first consequence of this covariance property is that $\rho_{out}(\vec{m})$ is invariant under rotation around the direction $\vec{m}$: $[e^{i\alpha\vec{m}\vec{\sigma}}\otimes e^{i\alpha\vec{m}\vec{\sigma}}, \rho_{out}(\vec{m})] = 0$ for all $\alpha$. This imposes conditions on the

$t_{jk}$ parameters. For example, if $\vec{m}$ is in the z-direction, then these conditions read: $t_{xx} = t_{yy}$, $t_{xy} = -t_{yx}$ and $t_{xz} = t_{zx} = t_{yz} = t_{zy} = 0$. Thus:

$$\begin{aligned}
\rho_{out}(\uparrow) &= \frac{1}{4}(\mathbb{1} + \eta(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) \qquad (3)\\
&+ t_{zz}\sigma_z \otimes \sigma_z + t_{xx}(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) + t_{xy}(\sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_x))\\
&= \frac{1}{4}\begin{pmatrix} 1+2\eta+t_{zz} & 0 & 0 & 0 \\ 0 & 1-t_{zz} & 2t_{xx}+2it_{xy} & 0 \\ 0 & 2t_{xx}-2it_{xy} & 1-t_{zz} & 0 \\ 0 & 0 & 0 & 1-2\eta+t_{zz} \end{pmatrix} \qquad (4)
\end{aligned}$$

Now, the no signaling condition implies [6]:

$$\rho_{out}(\uparrow) + \rho_{out}(\downarrow) = \rho_{out}(\rightarrow) + \rho_{out}(\leftarrow) \tag{5}$$

This implies that $t_{zz} = t_{xx} = t_{yy} \equiv t$.

Finally, the last condition that $\rho_{out}$ has to fulfil is positivity. The eigenvalues of $\rho_{out}(\uparrow)$ are:

$$\frac{1}{4}(1 \pm 2\eta + t) \tag{6}$$

$$\frac{1}{4}(1 - t \pm 2\sqrt{t^2 + t_{xy}^2}) \tag{7}$$

The maximum value of $\eta$ under the condition that these 4 eigenvalues are non-negative is obtained for $t_{xy} = 0$ and $t = \frac{1}{3}$. Hence $\eta_{max} = \frac{2}{3}$ and the maximum fidelity is given by:

$$\mathcal{F}_{max} = \frac{1 + \eta_{max}}{2} = \frac{5}{6} \tag{8}$$

This fidelity is obtained by the Bužek-Hillery QCM[3]. This proves that the Bužek-Hillery QCM is optimal, as already shown in [4, 5, 7].

A quite straightforward proof of the optimality of the Bužek-Hillery QCM[3] has been presented, based on the fact that no quantum process can provide arbitrary fast signaling[8]. Once again, quantum mechanics is right at the border line of contradicting relativity, but does not cross it. The peaceful coexistence between quantum mechanics and relativity[1] is thus reenforced. It is intriguing that the no signaling constraint is a powerful guide to find the limits of quantum mechanics[9].

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[6] The no signaling condition imposes that the mixtures of output states corresponding to indistinguishable mixtures of input states are themselves indistinguishable. This is more general than the condition used here. However, it turns out that in the present context, the simple condition (5) is sufficient.

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